REGULATOR OF MODULAR UNITS AND MAHLER MEASURES

WADIM ZUDILIN

ABSTRACT. We present a proof of the formula, due to Mellit and Brunault, which evaluates an integral of the regulator of two modular units to the value of the L-series of a modular form of weight 2 at $s = 2$. Applications of the formula to computing Mahler measures are discussed.

1. INTRODUCTION

The work of C. Deninger [6], D. Boyd [2], F. Rodriguez-Villegas [12] and others provided us with a natural link between the (logarithmic) Mahler measures

$$\mathrm{m}\big(P(x_1,\dots,x_m)\big) := \frac{1}{(2\pi i)^m}\int\cdots\int_{|x_1|=\dots=|x_m|=1}\log|P(x_1,\dots,x_m)|\,\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_m}{x_m}$$

of certain (Laurent) polynomials $P(x_1,\dots,x_m)$, higher regulators and Beĭlinson's conjectures, though it took a while for those original ideas to become proofs of some conjectural evaluations of Mahler measures. In this note we mainly discuss a recent general formula for the regulator of two modular units due to A. Mellit and F. Brunault, its consequences for 2-variable Mahler measures and some related problems.

For a smooth projective curve C given as the zero locus of a polynomial $P(x,y) \in \mathbb{C}[x,y]$ and two rational non-constant functions g and h on C, define the 1-form

$$\eta(g,h) := \log|g|\,\mathrm{d}\arg h - \log|h|\,\mathrm{d}\arg g; \tag{1}$$

here $\mathrm{d}\arg g$ is globally defined as $\mathrm{Im}(\mathrm{d}g/g)$. The form (1) is a real 1-form defined and infinitely many times differentiable on $C\setminus S$, where S is the set of zeros and poles of g and h. Furthermore, it is not hard to verify that the form (1) is antisymmetric, bi-additive and closed; the latter fact follows from

$$\mathrm{d}\eta(g,h) = \mathrm{Im}\bigg(\frac{\mathrm{d}g}{g}\wedge\frac{\mathrm{d}h}{h}\bigg) = 0,$$

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as the curve C has dimension 1. In turn, the closedness of (1) implies that, for a closed path γ in $C \setminus S$, the regulator map

$$r(\{g,h\})\colon \gamma \mapsto \int_\gamma \eta(g,h) \tag{2}$$

only depends on the homology class $[\gamma]$ of γ in $H_1(C \setminus S, \mathbb{Z})$.

Factorising $P(x,y)$ as a polynomial in y with coefficients from $\mathbb{C}[x]$,

$$P(x,y) = a_0(x)\prod_{j=1}^{n}(y - y_j(x)),$$

and applying Jensen's formula, we can write [5, 9, 12, 13] the Mahler measure of P in the form

$$\mathrm{m}\big(P(x,y)\big) = \mathrm{m}\big(a_0(x)\big) + \frac{1}{2\pi}\, r(\{x,y\})([\gamma]), \tag{3}$$

where

$$\gamma := \bigcup_{j=1}^{n}\big\{(x, y_j(x)) : |x| = 1,\ |y_j(x)| \ge 1\big\} = \{(x,y) \in C : |x| = 1,\ |y| \ge 1\} \tag{4}$$

is the union of at most n closed paths in $C \setminus S$.

In case the curve $C : P(x,y) = 0$ admits a parameterisation by means of modular *units* $x(\tau)$ and $y(\tau)$, where the modular parameter τ belongs to the upper halfplane $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$, one can change to the variable τ in the integral (2) for $r(\{x,y\})$; the class $[\gamma]$ in this case [4] becomes a union of paths joining certain cusps of the modular functions $x(\tau)$ and $y(\tau)$. The following general result completes the computation of the Mahler measure in the case when $x(\tau)$ and $y(\tau)$ are given as quotients/products of modular units

$$g_a(\tau) := q^{NB(a/N)/2} \prod_{\substack{n\ge1\\ n\equiv a \bmod N}}(1-q^n) \prod_{\substack{n\ge1\\ n\equiv -a \bmod N}}(1-q^n), \quad q = \exp(2\pi i\tau), \tag{5}$$
$$\text{where} \quad B(x) = B_2(x) := \{x\}^2 - \{x\} + \tfrac16.$$

Theorem 1 (Mellit–Brunault [11])**.** *For a, b and c integral, with ac and bc not divisible by N,*

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi}\, L(f(\tau) - f(i\infty), 2), \tag{6}$$

where the weight 2 modular form $f(\tau) = f_{a,b;c}(\tau)$ is given by

$$f_{a,b;c} := e_{a,bc}e_{b,-ac} - e_{a,-bc}e_{b,ac}$$

and

$$e_{a,b}(\tau) := \frac12\bigg(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1+\zeta_N^b}{1-\zeta_N^b}\bigg) + \sum_{m,n\ge1}(\zeta_N^{am+bn} - \zeta_N^{-(am+bn)})q^{mn}, \quad \zeta_N := \exp(2\pi i/N), \tag{7}$$

are weight 1 level N^2 Eisenstein series.

The L-value on the right-hand side of (6) is well defined because of subtracting the constant term

$$\begin{aligned} f(i\infty) &= \frac{1}{2}\biggl(\frac{1+\zeta_N^b}{1-\zeta_N^b}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}} - \frac{1+\zeta_N^a}{1-\zeta_N^a}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\biggr) \\ &= -\frac{1}{2}\biggl(\cot\frac{\pi b}{N}\,\cot\frac{\pi bc}{N} - \cot\frac{\pi a}{N}\,\cot\frac{\pi ac}{N}\biggr) \end{aligned}$$

in the q-expansion $f(\tau) = f(i\infty) + \sum_{n\ge1} c_n q^n$. Furthermore, if a linear combination

$$f(\tau) = \sum_{(a,b,c)\in\mathcal{M}} \lambda_{a,b,c} f_{a,b;c}(\tau), \qquad \lambda_{a,b,c}\in\mathbb{C},$$

happens to be a *cusp* form (and this corresponds to application of Theorem 1 to Mahler measures), then formula (6) produces the evaluation

$$\sum_{(a,b,c)\in\mathcal{M}} \lambda_{a,b,c} \int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi}\,L(f(\tau),2).$$

Note as well that the theorem allows one to integrate between any cusps c/N and d/N with the help of $\int_{c/N}^{d/N} = \int_{c/N}^{i\infty} - \int_{d/N}^{i\infty}$.

Here is a sketch of the proof of Theorem 1; details are given in Section 2. We parameterise the contour of integration by $\tau = c/N + it$, $0 < t < \infty$, and note that the Möbius transformation $\tau' := (c\tau - (c^2+1)/N)/(N\tau - c)$ preserves the contour: $\tau' = c/N + i/(N^2 t)$. Then the logarithms of $g_a(\tau)$ and $g_b(\tau)$, hence their real and imaginary parts — everything we need for computing the form (1), can be written as explicit Eisenstein series of weight 0 in powers of $\exp(-2\pi t)$ and $\exp(-2\pi/(N^2 t))$. Finally, executing the analytical change of variable from [14] the integrand becomes a linear combination of pairwise products of weight 1 Eisenstein series in powers of $\exp(-2\pi t)$ integrated against the form $t\,\mathrm{d}t$ along the line $0 < t < \infty$.

Applications of Theorem 1 to Boyd's and Rodriguez-Villegas' conjectural evaluations of 2-variable Mahler measures are discussed in Section 3, while Section 4 highlights some open problems related to 3-variable Mahler measures.

2. Proof of the Mellit–Brunault formula

The two auxiliary lemmas indicate particular modular transformations of the modular functions (5) and the Eisenstein series (7). Lemma 1 also describes the asymptotic behaviour of the modular functions (5) in a neighbourhood of a cusp with $\operatorname{Re}\tau = 0$; it is used in the form (10) to determine the integration contours (4) for our applications in Section 3.

Lemma 1. *For a,c integers,*

$$\begin{aligned}\log g_a(c/N+it) &= \pi icB(a/N)-\pi t\,NB(a/N)\\ &\qquad -\sum_{\substack{m,n\ge1\\ n\equiv a}}\frac{\zeta_N^{acm}}{m}\exp(-2\pi mnt)-\sum_{\substack{m,n\ge1\\ n\equiv -a}}\frac{\zeta_N^{-acm}}{m}\exp(-2\pi mnt)\\ &= -\frac{\pi i}{2}+\pi ia(c^2+1)(N-ac)+\pi icB(ac/N)-\frac{\pi B(ac/N)}{Nt}\\ &\qquad -\sum_{\substack{m,n\ge1\\ n\equiv ac}}\frac{\zeta_N^{-am}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)-\sum_{\substack{m,n\ge1\\ n\equiv -ac}}\frac{\zeta_N^{am}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr),\end{aligned}$$

where $t>0$.

Proof. First note that definition (5) implies

$$\begin{aligned}\log g_a(\tau) &= \pi i\tau\,NB(a/N)+\sum_{\substack{n\ge1\\ n\equiv a}}\log(1-q^n)+\sum_{\substack{n\ge1\\ n\equiv -a}}\log(1-q^n)\\ &= \pi i\tau\,NB(a/N)-\sum_{\substack{m,n\ge1\\ n\equiv a}}\frac{q^{mn}}{m}-\sum_{\substack{m,n\ge1\\ n\equiv -a}}\frac{q^{mn}}{m}.\end{aligned}$$

Therefore, the substitution $\tau=c/N+it$, equivalently $q=\zeta_N^c\exp(-2\pi t)$, results in the first expansion of the lemma.

Secondly, the modular units (5) are particular cases of the 'generalized Dedekind eta functions' [17, eq. (3)]. Applying [17, Theorem 1] with the choice $h=0$ and $\gamma=\left(\begin{smallmatrix} c & -c^2-1\\ 1 & -c\end{smallmatrix}\right)$ we deduce that

$$g_a(\tau)=\widetilde g_{a,c}\biggl(\frac{c\tau-(c^2+1)/N}{N\tau-c}\biggr),$$

where

$$\begin{aligned}\widetilde g_{a,c}(\tau) &:= \exp(-\pi i/2+\pi ia(c^2+1)(N-ac))\,q^{NB(ac/N)/2}\\ &\qquad\times\prod_{\substack{n\ge1\\ n\equiv ac \bmod N}}(1-\zeta_N^{-a(c^2+1)}q^n)\prod_{\substack{n\ge1\\ n\equiv -ac \bmod N}}(1-\zeta_N^{a(c^2+1)}q^n).\end{aligned}$$

On the other hand,

$$\tau':=\frac{c\tau-(c^2+1)/N}{N\tau-c}\bigg|_{\tau=c/N+it}=\frac{c}{N}+\frac{i}{N^2t},$$

so that

$$\begin{aligned}\log\widetilde g_{a,c}(\tau') &= -\frac{\pi i}{2}+\pi ia(c^2+1)(N-ac)+\pi icB(ac/N)-\frac{\pi B(ac/N)}{Nt}\\ &\quad -\sum_{\substack{m,n\ge1\\ n\equiv ac}}\frac{\zeta_N^{-a(c^2+1)m+cmn}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)-\sum_{\substack{m,n\ge1\\ n\equiv -ac}}\frac{\zeta_N^{a(c^2+1)m+cmn}}{m}\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr),\end{aligned}$$

and it remains to use the congruences $n \equiv ac$ and $n \equiv -ac$ to simplify the exponents of the roots of unity. ☐

Lemma 2. *For a, b integers not divisible by N,*

$$\frac{1}{N^2\tau}\, e_{a,b}\Big(-\frac{1}{N^2\tau}\Big) = \widetilde{e}_{a,b}(\tau) := \sum_{\substack{m,n\ge 1\\ m\equiv a,\ n\equiv b}} q^{mn} - \sum_{\substack{m,n\ge 1\\ m\equiv -a,\ n\equiv -b}} q^{mn}.$$

Proof. In [16, Section 7] the following general Eisenstein series of weight 1 and level N are introduced:

$$G_{a,c}(\tau) = G_{N,1;(c,a)}(\tau) := -\frac{2\pi i}{N}\Bigg(\kappa_{a,c} + \sum_{\substack{m,n\ge 1\\ n\equiv c \bmod N}} \zeta_N^{am} q^{mn/N} - \sum_{\substack{m,n\ge 1\\ n\equiv -c \bmod N}} \zeta_N^{-am} q^{mn/N}\Bigg),$$

where

$$\kappa_{a,c} := \begin{cases} \dfrac{1}{2}\,\dfrac{1+\zeta_N^a}{1-\zeta_N^a} & \text{if } c \equiv 0 \bmod N, \\ \dfrac{1}{2} - \left\{\dfrac{c}{N}\right\} & \text{if } c \not\equiv 0 \bmod N. \end{cases}$$

Then for $\gamma = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ we have

$$G_{a,c}(\gamma\tau) = (C\tau + D) G_{aD+cB,aC+cA}(\tau). \tag{8}$$

The partial Fourier transform from [7, Chapter III] applied to $G_{a,c}$ results in

$$\begin{aligned}\widehat{G}_{a,b}(\tau) := \sum_{c=0}^{N-1} \zeta_N^{bc}\, G_{a,c}(\tau) = -\frac{\pi i}{N}\Bigg(\frac{1+\zeta_N^a}{1-\zeta_N^a} + \frac{1+\zeta_N^b}{1-\zeta_N^b}\Bigg)\\ - \frac{2\pi i}{N}\sum_{m,n\ge 1} (\zeta_N^{am+bn} - \zeta_N^{-(am+bn)}) q^{mn/N}.\end{aligned}$$

On the other hand, taking $\gamma = \left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$ in (8) we find that

$$\begin{aligned}\tau^{-1}\widehat{G}_{a,b}(-1/\tau) &= \sum_{c=0}^{N-1} \zeta_N^{bc}\, G_{-c,a}(\tau)\\ &= -\frac{2\pi i}{N}\sum_{c=0}^{N-1} \zeta_N^{bc}\Bigg(\frac{1}{2} - \left\{\frac{a}{N}\right\} + \sum_{\substack{m,n\ge 1\\ n\equiv a}} \zeta_N^{-cm} q^{mn/N} - \sum_{\substack{m,n\ge 1\\ n\equiv -a}} \zeta_N^{cm} q^{mn/N}\Bigg)\\ &= -2\pi i\Bigg(\sum_{\substack{m,n\ge 1\\ n\equiv a,\ m\equiv b}} q^{mn/N} - \sum_{\substack{m,n\ge 1\\ n\equiv -a,\ m\equiv -b}} q^{mn/N}\Bigg).\end{aligned}$$

Using now $\widehat{G}_{a,b}(N\tau) = -2\pi i\, e_{a,b}(\tau)/N$ we obtain the desired transformation. ☐

The next two statements are to take care of integrating the constant terms of auxiliary Eisenstein series.

Lemma 3. *For a,b integers not divisible by N,*

$$\int_0^\infty\biggl(e_{a,b}(it)+e_{a,-b}(it)-\frac{1+\zeta_N^a}{1-\zeta_N^a}\biggr)\,t\,\mathrm{d}t=i\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)B\biggl(\frac{b}{N}\biggr),$$

where

$$\operatorname{Cl}_2(x):=\sum_{m\ge1}\frac{\sin mx}{m^2}$$

denotes Clausen's (dilogarithmic) function.

Proof. The integral under consideration is equal to

$$\begin{aligned}&\int_0^\infty\sum_{m,n\ge1}(\zeta_N^{am+bn}-\zeta_N^{-(am+bn)}+\zeta_N^{am-bn}-\zeta_N^{-(am-bn)})\exp(-2\pi mnt)\,t\,\mathrm{d}t\\&\quad=\int_0^\infty\sum_{m,n\ge1}(\zeta_N^{am}-\zeta_N^{-am})(\zeta_N^{bn}+\zeta_N^{-bn})\exp(-2\pi mnt)\,t\,\mathrm{d}t.\end{aligned}$$

On using the Mellin transform

$$\int_0^\infty\exp(-2\pi kt)t^{s-1}\,\mathrm{d}t=\frac{\Gamma(s)}{(2\pi)^sk^s}\qquad\text{for}\quad\operatorname{Re}s>0,\tag{9}$$

the integral of the double sum evaluates to

$$\frac{1}{4\pi^2}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m^2}\sum_{n\ge1}\frac{\zeta_N^{bn}+\zeta_N^{-bn}}{n^2}=\frac{i}{\pi^2}\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\sum_{n\ge1}\frac{\cos(2\pi nb/N)}{n^2}.$$

It remains to use

$$\sum_{n\ge1}\frac{\cos nx}{n^2}=\pi^2B\biggl(\frac{x}{2\pi}\biggr),$$

and the required evaluation follows. □

Lemma 4. *For a,b integers not divisible by N,*

$$\begin{aligned}&\int_0^\infty\frac{1}{iNt}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\biggl(\sum_{\substack{n\ge1\\n\equiv b}}-\sum_{\substack{n\ge1\\n\equiv-b}}\biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)\\&\quad=-i\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\frac{1+\zeta_N^b}{1-\zeta_N^b}.\end{aligned}$$

Proof. Performing the change of variable $u=1/(N^2t)$ in the integral, it becomes equal to

$$\frac{2\pi N}{i}\int_0^\infty\sum_{m\ge1}(\zeta_N^{am}-\zeta_N^{-am})\biggl(\sum_{\substack{n\ge1\\n\equiv b}}-\sum_{\substack{n\ge1\\n\equiv-b}}\biggr)n\exp(-2\pi mnu)\,u\,\mathrm{d}u,$$

and applying (9) with $s \to 2^+$ it evaluates to

$$\begin{aligned}
&\frac{N}{\pi}\sum_{m\ge1}\frac{\sin(2\pi am/N)}{m^2}\lim_{s\to1^+}\Biggl(\sum_{\substack{n\ge1\\ n\equiv b}}-\sum_{\substack{n\ge1\\ n\equiv -b}}\Biggr)\frac{1}{n^s}\\
&\quad=\frac{1}{\pi}\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\cdot\bigl(\psi(1-\{b/N\})-\psi(\{b/N\})\bigr)=\frac{1}{\pi}\operatorname{Cl}_2\biggl(\frac{2\pi a}{N}\biggr)\pi\cot\frac{\pi b}{N},
\end{aligned}$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. It remains to use $\cot(\pi b/N) = -i(1+\zeta_N^b)/(1-\zeta_N^b)$. $\square$

Proof of Theorem 1. To integrate the 1-form $\eta(g_a, g_b)$ along the interval $\tau \in (c/N, i\infty)$ we make the substitution $\tau = c/N + it$, $0 < t < \infty$. It follows from Lemma 1 that

$$\log|g_a(\tau)| = -\frac{\pi B(ac/N)}{Nt}-\frac{1}{2}\sum_{m\ge1}\frac{\zeta_N^{am}+\zeta_N^{-am}}{m}\Biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}+\sum_{\substack{n\ge1\\ n\equiv -ac}}\Biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr) \tag{10}$$

and

$$\begin{aligned}
\mathrm{d}\arg g_a(\tau) &= -\frac{1}{2i}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{acm}-\zeta_N^{-acm}}{m}\Biggl(\sum_{\substack{n\ge1\\ n\equiv a}}-\sum_{\substack{n\ge1\\ n\equiv -a}}\Biggr)\exp(-2\pi mnt)\\
&= \frac{1}{2i}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\Biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}-\sum_{\substack{n\ge1\\ n\equiv -ac}}\Biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr).
\end{aligned}$$

This computation implies

$$\begin{aligned}
\eta(g_a, g_b) = &-\frac{\pi B(ac/N)}{2iNt}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{bm}-\zeta_N^{-bm}}{m}\Biggl(\sum_{\substack{n\ge1\\ n\equiv bc}}-\sum_{\substack{n\ge1\\ n\equiv -bc}}\Biggr)\exp\biggl(-\frac{2\pi mn}{N^2t}\biggr)\\
&+\frac{1}{4i}\sum_{m_1\ge1}\frac{\zeta_N^{am_1}+\zeta_N^{-am_1}}{m_1}\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\Biggr)\exp\biggl(-\frac{2\pi m_1n_1}{N^2t}\biggr)\\
&\times\mathrm{d}\sum_{m_2\ge1}\frac{\zeta_N^{bcm_2}-\zeta_N^{-bcm_2}}{m_2}\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\Biggr)\exp(-2\pi m_2n_2t)
\end{aligned}$$

$$+\frac{\pi B(bc/N)}{2iNt}\,\mathrm{d}\sum_{m\ge1}\frac{\zeta_N^{am}-\zeta_N^{-am}}{m}\Biggl(\sum_{\substack{n\ge1\\ n\equiv ac}}-\sum_{\substack{n\ge1\\ n\equiv -ac}}\Biggr)\exp\Bigl(-\frac{2\pi mn}{N^2t}\Bigr)$$

$$-\frac{1}{4i}\sum_{m_1\ge1}\frac{\zeta_N^{bm_1}+\zeta_N^{-bm_1}}{m_1}\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\Biggr)\exp\Bigl(-\frac{2\pi m_1n_1}{N^2t}\Bigr)$$

$$\times\,\mathrm{d}\sum_{m_2\ge1}\frac{\zeta_N^{acm_2}-\zeta_N^{-acm_2}}{m_2}\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\Biggr)\exp(-2\pi m_2n_2t).$$

The terms involving double sums only can be integrated with the help of Lemma 4, and we obtain

$$\int_{c/N}^{i\infty}\eta(g_a,g_b)=\frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi b}{N}\Bigr)B\Bigl(\frac{ac}{N}\Bigr)-\frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi a}{N}\Bigr)B\Bigl(\frac{bc}{N}\Bigr)$$

$$-\frac{\pi}{2i}\Biggl(\sum_{m_1,m_2\ge1}(\zeta_N^{am_1}+\zeta_N^{-am_1})(\zeta_N^{bcm_2}-\zeta_N^{-bcm_2})\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\Biggr)\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\Biggr)$$

$$-\sum_{m_1,m_2\ge1}(\zeta_N^{bm_1}+\zeta_N^{-bm_1})(\zeta_N^{acm_2}-\zeta_N^{-acm_2})\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\Biggr)\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\Biggr)\Biggr)$$

$$\times\frac{n_2}{m_1}\int_0^\infty\exp\Bigl(-2\pi\Bigl(\frac{m_1n_1}{N^2t}+m_2n_2t\Bigr)\Bigr)\,\mathrm{d}t.$$

Now we execute the change of variable $u=n_2t/m_1$, interchange integration and quadruple summation and use Lemma 2:

$$\int_{c/N}^{i\infty}\eta(g_a,g_b)=\frac{\pi i}{2}\,\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi b}{N}\Bigr)B\Bigl(\frac{ac}{N}\Bigr)-\frac{\pi i}{2}\,\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\,\mathrm{Cl}_2\Bigl(\frac{2\pi a}{N}\Bigr)B\Bigl(\frac{bc}{N}\Bigr)$$

$$-\frac{\pi}{2i}\int_0^\infty\sum_{m_1,m_2\ge1}(\zeta_N^{am_1}+\zeta_N^{-am_1})(\zeta_N^{bcm_2}-\zeta_N^{-bcm_2})\exp(-2\pi m_1m_2u)$$

$$\times\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv ac}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -ac}}\Biggr)\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv b}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -b}}\Biggr)\exp\Bigl(-\frac{2\pi n_1n_2}{N^2u}\Bigr)$$

$$-\sum_{m_1,m_2\ge1}(\zeta_N^{bm_1}+\zeta_N^{-bm_1})(\zeta_N^{acm_2}-\zeta_N^{-acm_2})\exp(-2\pi m_1m_2u)$$

$$\times\Biggl(\sum_{\substack{n_1\ge1\\ n_1\equiv bc}}+\sum_{\substack{n_1\ge1\\ n_1\equiv -bc}}\Biggr)\Biggl(\sum_{\substack{n_2\ge1\\ n_2\equiv a}}-\sum_{\substack{n_2\ge1\\ n_2\equiv -a}}\Biggr)\exp\Bigl(-\frac{2\pi n_1n_2}{N^2u}\Bigr)\,\mathrm{d}u$$

$$
\begin{aligned}
&= \frac{\pi i}{2}\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\operatorname{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\operatorname{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big)\\
&\quad - \frac{\pi}{2i}\int_0^\infty \Big(e_{a,bc}(iu) - e_{a,-bc}(iu) - \frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\big(\widetilde{e}_{b,ac}(i/(N^2u)) + \widetilde{e}_{b,-ac}(i/(N^2u))\big)\\
&\qquad - \Big(e_{b,ac}(iu) - e_{b,-ac}(iu) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\Big)\big(\widetilde{e}_{a,bc}(i/(N^2u)) + \widetilde{e}_{a,-bc}(i/(N^2u))\big)\,\mathrm{d}u\\
&= \frac{\pi i}{2}\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\operatorname{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\operatorname{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big)\\
&\quad + \frac{\pi}{2}\int_0^\infty \Big(e_{a,bc}(iu) - e_{a,-bc}(iu) - \frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\big(e_{b,ac}(iu) + e_{b,-ac}(iu)\big)\,u\\
&\qquad - \Big(e_{b,ac}(iu) - e_{b,-ac}(iu) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\Big)\big(e_{a,bc}(iu) + e_{a,-bc}(iu)\big)\,u\,\mathrm{d}u\\
&= \frac{\pi i}{2}\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\operatorname{Cl}_2\Big(\frac{2\pi b}{N}\Big)B\Big(\frac{ac}{N}\Big) - \frac{\pi i}{2}\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\operatorname{Cl}_2\Big(\frac{2\pi a}{N}\Big)B\Big(\frac{bc}{N}\Big)\\
&\quad + \pi\int_0^\infty \big(e_{a,bc}(iu)e_{b,-ac}(iu) - e_{a,-bc}(iu)e_{b,ac}(iu)\big)\,u\\
&\qquad - \frac{1}{2}\Big(\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\big(e_{b,ac}(iu) + e_{b,-ac}(iu)\big) - \frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}}\big(e_{a,bc}(iu) + e_{a,-bc}(iu)\big)\Big)u\,\mathrm{d}u
\end{aligned}
$$

(we apply Lemma 3)

$$
= \pi\int_0^\infty \Big(f_{a,b;c}(iu) + \frac{1}{2}\frac{1+\zeta_N^{a}}{1-\zeta_N^{a}}\frac{1+\zeta_N^{ac}}{1-\zeta_N^{ac}} - \frac{1}{2}\frac{1+\zeta_N^{b}}{1-\zeta_N^{b}}\frac{1+\zeta_N^{bc}}{1-\zeta_N^{bc}}\Big)\,u\,\mathrm{d}u,
$$

and the result follows by appealing to (9). □

3. Applications

The modularity theorem guarantees that an *elliptic* curve $C : P(x, y) = 0$ can be parameterised by modular functions $x(\tau)$ and $y(\tau)$, whose level N is necessarily the conductor of C, such that the pull-back of the canonical differential on C is proportional to $2\pi i f(\tau)\,\mathrm{d}\tau = f(\tau)\,\mathrm{d}q/q$, where f is (up to an isogeny) a normalised newform of weight 2 and level N, which automatically happens to be a cusp form and a Hecke eigenform. Computing the conductor of C and producing the cusp form f of this level give one an efficient strategy to determine successively the coefficients in the q-expansions of $x(\tau) = \varepsilon_1 q^{-M_1} + \cdots$ and $y(\tau) = \varepsilon_2 q^{-M_2} + \cdots$ subject to $P(x(\tau), y(\tau)) = 0$, where ε_1 and ε_2 are suitable nonzero constants. The particular form of q-expansions only fixes a normalisation of $x(\tau)$ and $y(\tau)$ up to the action of the corresponding congruence subgroup $\Gamma_0(N)$. Finally, it remains to verify whether $x(\tau)$ and $y(\tau)$ just found are modular units — modular functions whose all zeroes and poles are at cusps (so that they admit eta-like product expansions); if this is the case, we can use Theorem 1 to compute the Mahler measure $\mathrm{m}(P(x, y))$.

In this section we touch the 'classical' family of Mahler measures

$$\mathrm{m}(xy^2+(x^2+kx+1)y+x)=\mathrm{m}\Big(k+x+\frac{1}{x}+y+\frac{1}{y}\Big), \qquad k^2\in\mathbb{Z}\setminus\{0,16\},$$

which goes back to the works [2, 6, 12]. Namely, we will see that Theorem 1 applies in the cases when the corresponding zero locus

$$E: k+x+\frac{1}{x}+y+\frac{1}{y}=0 \tag{11}$$

can be parameterised by modular units. For this family, equation (3) assumes the form

$$\mathrm{m}\Big(k+x+\frac{1}{x}+y+\frac{1}{y}\Big)=\mathrm{m}(y^2+(k+x+x^{-1})y+1)=\frac{1}{2\pi}r(\{x,y\})([\gamma]), \tag{12}$$

where γ is a single closed path on $E\setminus\{(0,0)\}$ corresponding to the zero $y_1(x)$ of $y^2+(k+x+x^{-1})y+1$ which satisfies $|y_1(x)|\ge 1$.

The above general strategy restricted to the family (11) was identified by Mellit in [10] and illustrated by him on the example of $k=2i$; this is Example 2 below. The modular functions x and y satisfying (11) are searched in the form $x(\tau)=(\varepsilon q)^{-1}+\cdots$ and $y(\tau)=-(\varepsilon q)^{-1}+\cdots$, where $\varepsilon\in\mathbb{Z}[k]$ is chosen so that k/ε is a positive integer. The condition on the pull-back of the canonical differential on E takes the form

$$\frac{q\,(\mathrm{d}x/\mathrm{d}q)}{\varepsilon x(y-1/y)}=f,$$

where $f(\tau)$ is the corresponding Hecke eigenform of weight 2.

The computational part of the examples below was accomplished in `sage` and `gp-pari`. Below we will have occasional appearance of Dedekind's eta-function $\eta(\tau):=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. We hope that this extra eta notation does not cause any confusion with (1), as it is here depends on a *single* variable, which is always a rational multiple of τ from the upper halfplane.

Example 1. The most classical example corresponds to the choice $k=1$, when the elliptic curve in (11) has conductor $N=15$ and can be parameterised by modular units

$$\begin{aligned} x(\tau)&=\frac{1}{q}\prod_{n=0}^{\infty}\frac{(1-q^{15n+7})(1-q^{15n+8})}{(1-q^{15n+2})(1-q^{15n+13})}=\frac{g_7(\tau)}{g_2(\tau)},\\ y(\tau)&=-\frac{1}{q}\prod_{n=0}^{\infty}\frac{(1-q^{15n+4})(1-q^{15n+11})}{(1-q^{15n+1})(1-q^{15n+14})}=-\frac{g_4(\tau)}{g_1(\tau)}, \end{aligned}$$

so that

$$\frac{q\,(\mathrm{d}x/\mathrm{d}q)}{x(y-1/y)}=f_{15}(\tau):=\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$$

and the path of integration γ in (12) corresponds to the range of τ between the two cusps $-1/5$ and $1/5$ of $\Gamma_0(15)$. Therefore, Theorem 1 results in

$$\begin{aligned}
\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= \frac{1}{2\pi}\Big(\int_{-1/5}^{i\infty}-\int_{1/5}^{i\infty}\Big)\eta(g_7/g_2, g_4/g_1) \\
&= \frac{1}{8\pi^2}L(2f_{7,4;-3}-2f_{7,1;-3}-2f_{2,4;-3}+2f_{2,1;-3},2) \\
&= \frac{15}{4\pi^2}L(f_{15},2),
\end{aligned}$$

which is precisely Boyd's conjecture from [2] first proven in [15].

Note that this evaluation implies some other Mahler measures, namely [8, 9]

$$\begin{aligned}
\mathrm{m}\Big(5+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 6\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big) \\
\mathrm{m}\Big(16+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 11\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big), \\
\mathrm{m}\Big(3i+x+\frac{1}{x}+y+\frac{1}{y}\Big) &= 5\mathrm{m}\Big(1+x+\frac{1}{x}+y+\frac{1}{y}\Big),
\end{aligned}$$

though the corresponding elliptic curves $k+x+1/x+y+1/y=0$ for $k=5$, 16 and $3i$ are not parameterised by modular units.

Example 2 ([10])**.** The modular parameterisation of (11) for $k=2i$ (the conductor of elliptic curve is then $N=40$) and the corresponding Mahler measure evaluation

$$\mathrm{m}\Big(2i+x+\frac{1}{x}+y+\frac{1}{y}\Big) = \frac{10}{\pi^2}L(f_{40},2),$$

where

$$f_{40}(\tau) := \frac{\eta(\tau)\eta(8\tau)\eta(10\tau)^2\eta(20\tau)^2}{\eta(5\tau)\eta(40\tau)} + \frac{\eta(2\tau)^2\eta(4\tau)^2\eta(5\tau)\eta(40\tau)}{\eta(\tau)\eta(8\tau)},$$

were given in Mellit's talk [10]. He identifies $x(\tau)$ and $y(\tau)$ with infinite products which are fully expressible by means of Ramanujan's lambda function

$$\lambda(\tau) = q^{1/5}\prod_{n=1}^{\infty}(1-q^n)^{\left(\frac{n}{5}\right)} = q^{1/5}\prod_{n=1}^{\infty}\frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})};$$

namely,

$$\begin{aligned}
x(\tau) &= -i\,\frac{\lambda(4\tau)}{\lambda(\tau)\lambda(8\tau)} = -i\,\frac{g_2g_3g_7g_{13}g_{16}g_{17}g_{18}}{g_1g_6g_8g_9g_{11}g_{14}g_{19}}, \\
y(\tau) &= i\,\frac{\lambda(\tau)\lambda(2\tau)}{\lambda(8\tau)} = i\,\frac{g_1g_9g_{11}g_{16}g_{19}}{g_3g_7g_8g_{13}g_{17}}
\end{aligned}$$

in the notation (5) with $N=40$. The corresponding range of τ for the path γ in (12) is from $1/10$ to $-2/5$.

Example 3. The elliptic curve (11) for $k=2$ has conductor $N=24$ and admits parameterisation by modular units

$$x(\tau)=\frac{g_1g_{10}g_{11}}{g_2g_5g_7}, \qquad y(\tau)=-\frac{g_5g_7}{g_1g_{11}}.$$

Theorem 1 applies and produces the evaluation

$$\begin{aligned}\mathrm{m}\Big(2+x+\frac{1}{x}+y+\frac{1}{y}\Big)&=\frac{1}{2\pi}\Big(\int_{-1/8}^{i\infty}-\int_{1/8}^{i\infty}\Big)\eta\Big(\frac{g_1g_{10}g_{11}}{g_2g_5g_7},\frac{g_5g_7}{g_1g_{11}}\Big)\\&=\frac{6}{\pi^2}L(f_{24},2),\end{aligned}$$

where $f_{24}(\tau):=\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$, conjectured in [2] and established in [14]. Note that another curve (11) with $k=8$ of the same conductor $N=24$ can be parameterised by modular units as well: the pair

$$x(\tau)=\Big(\frac{g_1g_5g_7g_{11}}{g_4}\Big)^4, \qquad y(\tau)=-\Big(\frac{g_2g_{10}}{g_1g_4g_5g_7g_{11}}\Big)^4$$

satisfies $8+x+1/x+y+1/y=0$; however, it is a subtle problem to fix the integration path γ for this parameterisation. Note that

$$\mathrm{m}\Big(8+x+\frac{1}{x}+y+\frac{1}{y}\Big)=4\mathrm{m}\Big(2+x+\frac{1}{x}+y+\frac{1}{y}\Big)$$

is already known [9].

Example 4. For $N=17$, the pair of modular units

$$x(\tau)=-i\,\frac{g_2g_8}{g_1g_4}, \qquad y(\tau)=i\,\frac{g_6g_7}{g_3g_5}$$

parameterise the elliptic curve $i+x+1/x+y+1/y=0$. Applying Theorem 1 for τ ranging from $3/17$ to $-3/17$, we obtain

$$\mathrm{m}\Big(i+x+\frac{1}{x}+y+\frac{1}{y}\Big)=\frac{17}{2\pi^2}L(f_{17},2),$$

where

$$\begin{aligned}f_{17}(\tau):=\frac{q\,(\mathrm{d}x/\mathrm{d}q)}{ix(y-1/y)}&=q-q^2-q^4-2q^5+4q^7+3q^8-3q^9+2q^{10}\\&\quad-2q^{13}-4q^{14}-q^{16}+q^{17}+O(q^{18}).\end{aligned}$$

This Mahler measure evaluation was conjectured in [12, Table 4].

Example 5. Another conjecture in [12, Table 4],

$$\mathrm{m}\Big(\sqrt{2}+x+\frac{1}{x}+y+\frac{1}{y}\Big)=\frac{7}{2\pi^2}L(f_{56},2),$$

corresponds to $k=\sqrt{2}$ in (11) and an elliptic curve over $\mathbb{Z}$ of conductor $N=56$. It is parameterised by the couple

$$\begin{aligned} x(\tau) &= \frac{1}{\sqrt{2}}\,\frac{\eta(\tau)\eta(4\tau)^2\eta(7\tau)\eta(28\tau)^2}{\eta(2\tau)^2\eta(8\tau)\eta(14\tau)^2\eta(56\tau)}, \\ y(\tau) &= -\frac{1}{\sqrt{2}}\,\frac{\eta(2\tau)\eta(4\tau)\eta(14\tau)\eta(28\tau)}{\eta(\tau)\eta(7\tau)\eta(8\tau)\eta(56\tau)}, \end{aligned}$$

so that

$$\begin{aligned} f_{56}(\tau) := \frac{q\,(\mathrm{d}x/\mathrm{d}q)}{\sqrt{2}\,x(y-1/y)} &= q+2q^5-q^7-3q^9-4q^{11}+2q^{13}-6q^{17}+8q^{19} \\ &\quad -q^{25}+6q^{29}+8q^{31}+O(q^{34}). \end{aligned}$$

It is not clear whether there are finitely or infinitely many cases of the parameter k in (11) subject to parameterisation by modular units. A possible approach in cases when such parameterisation is not available is writing down algebraic relations between any two standard modular units (5) of a given level N and sieving the relations which may be used in producing the Mahler measures of 2-variable polynomials which are potentially linked to the wanted Mahler measures by K-theoretic machinery [5, 8, 9].

Finding what curves $C: P(x,y)=0$ can be parameterised by modular units is an interesting question itself. F. Brunault notices some heuristics to the fact that there are only finitely many function fields F of a given genus g over $\mathbb{Q}$ which embed into the function field of a modular curve such that F can be generated by modular units; for $g\ge 2$ this follows from [1, Conjecture 1.1]. In fact, he recently studied the following related question: find all the elliptic curves E over $\mathbb{Q}$ whose canonical parameterisation $\varphi\colon X_1(N)\to E$ is such that the pre-image of the rational torsion subgroup consists only of cusps. Brunault shows that there are only finitely many elliptic curves with this property and produces the list of all them.

4. 3-variable Mahler measures

It would be desirable to have an analogue of Theorem 1 for 3-variable Mahler measures of (Laurent) polynomials $P(x,y,z)$ such that the intersection of the zero loci $P(x,y,z)=0$ and $P(1/x,1/y,1/z)=0$ defines an elliptic curve E, and $\mathrm{m}(P)$ is presumably related to the L-series of E evaluated at $s=3$. No example of this type is established, and one of the simplest evaluations is Boyd's conjecture [3]

$$\mathrm{m}\big((1+x)(1+y)-z\big) \overset{?}{=} 2L'(E_{15},-1)=\frac{225}{4\pi^4}L(E_{15},3).$$

On the surface $(1+x)(1+y)-z=0$ we have

$$\begin{aligned} x\wedge y\wedge z &= x\wedge y\wedge(1+x)(1+y) = x\wedge y\wedge(1+x)+x\wedge y\wedge(1+y) \\ &= -x\wedge(1+x)\wedge y + y\wedge(1+y)\wedge x \\ &= -(-x)\wedge(1+x)\wedge y + (-y)\wedge(1+y)\wedge x. \end{aligned}$$

Applying the machinery described in [5, Section 5.2] to the 3-variable polynomial $P(x,y,z)=(1+x)(1+y)-z$ we obtain

$$m(P)=\frac{1}{4\pi^2}\int_\gamma\big(\omega(-x,y)-\omega(-y,x)\big),$$

where

$$\omega(g,h):=D(g)\,\mathrm{d}\arg h+\frac{1}{3}\big(\log|g|\,\mathrm{d}\log|1-g|-\log|1-g|\,\mathrm{d}\log|g|\big)\log|h| \qquad (13)$$

and

$$\begin{aligned}\gamma:=\{(x,y,z):|x|=|y|=|z|=1\}&\cap\{(x,y,z):(1+x)(1+y)-z=0\}\\ &\cap\{(x,y,z):(1+x)(1+y)z-xy=0\}.\end{aligned}$$

Note that $\{(1+x)(1+y)-z=0\}\cap\{(1+x)(1+y)z-xy=0\}$ is the double cover of an elliptic curve of conductor 15. Indeed, eliminating z we can write (one half of) its equation as

$$(1+x_1^2)(1+y_1^2)+x_1y_1=0$$

in variables $x_1=\sqrt{x}$, $y_1=\sqrt{y}$, or

$$x_2+1/x_2+y_2+1/y_2+1=0$$

in variables $x_2=x_1y_1$, $y_2=x_1/y_1$. Using the parameterisation of the latter equation by the modular units from Example 1 we find out that

$$m(P)=\frac{1}{2\pi^2}\int_{-1/5}^{1/5}\big(\omega(X,Y)-\omega(Y,X)\big)$$

where

$$X(\tau):=\frac{g_4(\tau)g_7(\tau)}{g_1(\tau)g_2(\tau)}=q^{-2}+O(q^{-1}) \quad\text{and}\quad Y(\tau):=\frac{g_1(\tau)g_7(\tau)}{g_2(\tau)g_4(\tau)}=1+O(q).$$

Also note that

$$1-X(\tau)=-\frac{g_6(\tau)g_7(\tau)}{g_1(\tau)g_3(\tau)}=-q^{-2}+O(q^{-1}) \quad\text{and}\quad 1-Y(\tau)=\frac{g_1(\tau)g_3(\tau)}{g_2(\tau)g_6(\tau)}=q+O(q^2)$$

are modular units.

The problem with integrating the form (13) is that it is, roughly speaking, integrating the product of *three* modular components: two of them are logarithms of modular functions (hence of weight 0) and one is the logarithmic derivative of a modular function (hence of weight 2). On the other hand, the expected data for applying the method from [14] used in our proof of Theorem 1 in Section 2 would be integrating a product of *two* Eisenstein series of weights -1 and 3 (see [18] for details).

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School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, AUSTRALIA

E-mail address: `wzudilin@gmail.com`